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ARTICLE

TENSION SPLINE SOLUTIONS FOR FOURTH ORDER SINGULARLYPERTURBED BOUNDARY-VALUE PROBLEMS

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ABSTRACT

We use tension spline to develop numerical methods for the solution singularly perturbed boundary-value problems. The proposed methods are accurate for solution of fourth order boundary-value problems. End conditions of the spline are derived. Two examples are considered for the numerical illustration. However, it is observed that our approach produce better numerical solutions in the sense thatmax$|e_i|$is minimum.

KEY WORDS
Tension spline, Singularlyperturbed boundary-valueproblems, Boundaryformulae.

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INTRODUCTION

We consider fourth-order boundary value problem of type:

$$\varepsilon u^{(4)}(x) = f(x,u), \quad x \in [a,b] \tag{1}$$

with boundary conditions

$$u(a) = \sigma_1 ,\ y(b) = \sigma_2 ,\ y^{(2)}(a) = \sigma_3 ,\ y^{(2)}(b) = \sigma_4 \tag{2}$$

where σ_i for $i = 1,2,3,4$ are finite real constants and the functions $u(x)$ and $f(x,u)$ are continuous on $[a,b]$ and ε is a parameter such that $0 < \varepsilon < 1$.

Several methods such as an asymptotic finite element method, quintic B-spline method, initial value techniques, differential transform method, variable mesh difference methods, fourth-order spline method, Non-polynomial sextic spline and tension splines, in solving singularly perturbed boundary-value problems has been of considerable concern and is well covered in papers see [1]-[10]. Akram et al. [11]-[12] used septic spline and quintic spline for the solution of fourth order singularly perturbed boundary value problem.

Following the spline functions proposed in this paper have the form $T_9 = \text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, \cos(kx), \sin(kx)\}$ where k is the frequency of the trigonometric part of the spline functions. Thus in each subinterval $x_i \le x \le x_{i+1}$ we have

$$\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9\}, \text{ (When } k \to 0 \text{)}$$

In this paper, we use tension spline approximation to develop a family of new numerical methods to obtain smooth approximations to the solution of singularly perturbed boundary-value problems. In Section 2, the new tension spline methods are developed for solving Eq. (1) along with boundary condition Eq. (2). and also development of the boundary formulas. Section 3, tension spline solution of (1) and (2) is determined and in section 4, numerical experiment, discussion are given.

NUMERICAL METHODS

To develop the spline approximation to the boundary-value problem Eqs. (1)-(2), the interval $[a,b]$ is divided in to n equal subintervals using the grid

$$x_0 = a,\ x_i = a + ih\ ,\ h = \frac{b-a}{n},\ i = 0,1,2,..,n,\ \ x_n = b.$$

We consider the following tension spline $S_i(x)$ on each subinterval $[x_i, x_{i+1}]$, $i = 0,1,2,..,n-1$,

$$S_i(x) = a_i \cosh k(x - x_i) + b_i \sinh k(x - x_i) + c_i(x - x_i)^7 + d_i(x - x_i)^6 + e_i(x - x_i)^5 + o_i(x - x_i)^4 + p_i(x - x_i)^3 + q_i(x - x_i)^2 + r_i(x - x_i) + y_i \tag{3}$$

where $a_i, b_i, c_i, d_i, e_i, o_i, p_i, r_i$ and y_i are real finite coefficients and k is arbitrary parameterwhich have to be determine so that, the spline is defined in terms of its 2th, 4th, 6th and 8th derivatives and we denote these values at knots as:

$$S_i(x_l) = u_l\ , S_i^{''}(x_l) = m_l\ ,\ S_i^{(4)}(x_l) = z_l,\ S_i^{(6)}(x_l) = v_l,\ S_i^{(8)}(x_l) = p_l,$$
$$\text{for } i = 0,1,2,...,n-1. \text{ and } l = i, i+1. \tag{4}$$

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Assuming $u(x)$ to be the function which has be interpolated by $S_i(x)$, and $u(x_i)$ be an approximation to $u(x_i)$, using the continuity conditions of first, three, fifth and seventh ($S_{i-1}^{(\mu)}(x_i)=S_i^{(\mu)}(x_i)$ where $\mu=1,3,5$ and 7), and also by eliminating of m_i, v_i and p_i, we obtain the following relations between u_i and z_i :

$$\alpha_1 z_{i-4}+\alpha_2 z_{i-3}+\alpha_3 z_{i-2}+\alpha_4 z_{i-1}+\alpha_5 z_i+\alpha_4 z_{i+1}+\alpha_3 z_{i+2}+\alpha_2 z_{i+3}+\alpha_1 z_{i+4}=$$
$$-\frac{1}{h^4}(\beta_1 u_{i-4}+\beta_2 u_{i-3}+\beta_3 u_{i-2}+\beta_4 u_{i-1}+\beta_5 u_i+\beta_4 u_{i+1}+\beta_3 u_{i+2}+\beta_2 u_{i+3}+\beta_1 u_{i+4}),$$
$$i=4,5,,\ldots,n-4. \qquad (5)$$

where

$$\alpha_1=\frac{1}{\gamma_1}(\gamma_3-7!\sinh(\theta)),\quad \alpha_2=\frac{-2}{\gamma_1}(\gamma_3\cosh(\theta)-12(-1260\theta+42\theta^5+5\theta^7+1680\sinh(\theta)),$$

$$\alpha_3=\frac{-8}{\gamma_1}(-10080\theta+840\theta^3-84\theta^5-149\theta^7+6\theta(-1260+42\theta^4+5\theta^6)\cosh(\theta)+17640\sinh(\theta)),$$

$$\alpha_4=\frac{-2}{\gamma_1}(\gamma_2\cosh(\theta)-4(-16380\theta+1680\theta^3-294\theta^5+317\theta^7+35280\sinh(\theta)),$$

$$\alpha_5=\frac{2}{\gamma_1}(\gamma_2-16\theta(-6300+840\theta^2-210\theta^4+151\theta 6)\cosh(\theta)-176400\sinh(\theta)),$$

$$\beta_1=\frac{840\theta^4}{\gamma_1}(6\theta+\theta^3 6\sinh(\theta)),\ \beta_2=\frac{1680\theta^4}{\gamma_1}(\theta(6+\theta^2)\cosh(\theta)+6(3\theta-4\sinh(\theta))),$$

$$\beta_3=\frac{6720\theta^4}{\gamma_1}(-12\theta+\theta^3-9\theta\cosh(\theta)+21\sinh(\theta)),\ \beta_4=\frac{-1680\theta^4}{\gamma_1}(-78\theta+8\theta^3+9\theta(-10+\theta^2)\cosh(\theta)+168\sinh(\theta)),$$

$$\beta_5=\frac{1680\theta^4}{\gamma_1}(-90\theta+9\theta^3+8\theta(-15+2\theta^2)\cosh(\theta)+210\sinh(\theta)),\ \gamma_1=2\theta(420+70\theta^2+3\theta^4)+(-840+\theta^4)\sinh(\theta),$$

$$\gamma_2=75600\theta-7560\theta^3+630\theta^5+1191\theta^7),\ \gamma_3=7!\theta+840\theta^3+42\theta^5+\theta^7,$$

where $\theta=kh$.
We assume that

$$\varepsilon u_i^{(4)}=f(x_i,u_i)=f_i\equiv f(x_i,u(x_i)), \qquad (6)$$

where f is nonlinear with respect to u and u_i is the approximation of the exact value $u(x_i)$ and $S_i(x)$ is tensionspline function.
Now the local truncation error corresponding to the tensionspline method (5) can be obtain as:

$$t_i=(2(\beta_1+\beta_2+\beta_3+\beta_4)+\beta_5)u_i+(16\beta_1+9\beta_2+4\beta_3+\beta_4)h^2u_i^{(2)}$$
$$+(2(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+\frac{1}{12}(256\beta_1+81\beta_2+16\beta_3+\beta_4))h^4u_i^{(4)}$$
$$+((16\alpha_1+9\alpha_2+4\alpha_3+\alpha_4)+\frac{2}{6!}(4096\beta_1+729\beta_2+64\beta_3+\beta_4))h^6u_i^{(6)}$$
$$+\frac{2}{8!}(1680(256\alpha_1+81\alpha_2+16\alpha_3+\alpha_4)+65536\beta_1+6561\beta_2+256\beta_3+\beta_4)h^8u_i^{(8)}$$
$$+\frac{2}{10!}(5040(4096\alpha_1+729\alpha_2+64\alpha_3+\alpha_4)+1048576\beta_1+59049\beta_2+1024\beta_3+\beta_4)h^{10}u_i^{(10)}$$
$$+\frac{2}{12!}(11880(65536\alpha_1+6561\alpha_2+256\alpha_3+\alpha_4)+16777216\beta_1+531441\beta_2+4096\beta_3+\beta_4)h^{12}u_i^{(12)}$$
$$+\frac{2}{14!}(48048(1048576\alpha_1+59049\alpha_2+1024\alpha_3+\alpha_4)+536870912\beta_1+9565938\beta_2+32768\beta_3+\beta_4)h^{14}u_i^{(14)}$$
$$+\frac{1}{16!}(43680(16777216\alpha_1+531441\alpha_2+4096\alpha_3+\alpha_4)+4294967296\beta_1+43046721\beta_2+65536\beta_3+\beta_4)h^{16}u_i^{(16)}$$
$$+\frac{1}{18!}(73440(268435456\alpha_1+4782969\alpha_2+16384\alpha_3+\alpha_4)+68719476736\beta_1+387420489\beta_2+262144\beta_3+\beta_4)h^{18}u_i^{(18)}+\ldots$$
$$(7)\ i=6,7,\ldots,n-6$$

for different choices of parameters $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5$ we can obtain the following classes of methods such as:

Case(1): By choosing $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0,\ \alpha_5=-1,\beta_1=\beta_2=0,\beta_3=1,\beta_4=-4$ and $\beta_5=6$ we obtain the second-order method with truncation error $t_i=\frac{1}{6}h^6u_i^{(6)}+O(h^8)$.

Case(2): By choosing $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0,\ \alpha_5=6,\beta_1=0,\beta_2=1,\beta_3=-12,\beta_4=39$ and $\beta_5=-56$ we obtain the fourth-order method with truncation error $t_i=\frac{7}{40}h^8u_i^{(8)}+O(h^{10})$.

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Case(3): By choosing

$\alpha_1=-\frac{1}{3024},\alpha_2=-\frac{502}{3024},\alpha_3=-\frac{14608}{3024},\alpha_4=-\frac{88234}{3024},\ \alpha_5=-\frac{156190}{3024},\beta_1=1,\beta_2=22,\beta_3=-32,\beta_4=-86$ and $\beta_5=190$ we obtain the sixth-order method with truncation error $t_i=-\frac{1}{252}h^{10}u_i^{(10)}+O(h^{12})$.

Case(4): By choosing

$\alpha_1=-\frac{1}{3024},\alpha_2=-\frac{35}{216},\alpha_3=-\frac{1835}{378},\alpha_4=-\frac{44027}{1512},\ \alpha_5=-\frac{78215}{1512},\beta_1=1,\beta_2=22,\beta_3=-32,\beta_4=-86$ and $\beta_5=190$ we obtain the eighth-order method with truncation error $t_i=\frac{43}{30240}h^{12}u_i^{(12)}+O(h^{14})$.

Case(5): By choosing

$\alpha_1=-\frac{53}{30240},\alpha_2=-\frac{1139}{7560},\alpha_3=-\frac{37001}{7560},\alpha_4=-\frac{219533}{7560},\ \alpha_5=-\frac{156731}{3024},\beta_1=1,\beta_2=22,\beta_3=-32,\beta_4=-86$ and $\beta_5=190$ we obtain the tenth-order method with truncation error $t_i=-\frac{107}{798336}h^{14}u_i^{(14)}+O(h^{16})$.

Case(6): By choosing

$\alpha_1=\frac{772159}{415638720},\alpha_2=-\frac{10411993}{25977420},\alpha_3=-\frac{663600673}{103909680},\alpha_4=-\frac{814474831}{25977420},\ \alpha_5=-\frac{2269987033}{41563872}$, and $\beta_1=\frac{52717}{20617},\beta_2=\frac{376534}{20617},\beta_3=\frac{-614804}{20617},\beta_4=-86$ and $\beta_5=190$ we obtain the twelve-order method with truncation error $t_i=-\frac{22493389}{5446529786880}h^{16}u_i^{(16)}+O(h^{18})$.

Case(7): By choosing

$\alpha_1=\frac{-94699819}{26826813405},\alpha_2=-\frac{22043001088}{26826813405},\alpha_3=-\frac{41159857396}{3832401915},\alpha_4=-\frac{171489887488}{3832401915},\ \alpha_5=-\frac{55281435446}{766480383}$, and $\beta_1=\frac{448760563}{85164487},\beta_2=\frac{1891726336}{85164487},\beta_3=\frac{-4591530956}{85164487},\beta_4=-\frac{5839582208}{85164487}$ and $\beta_5=190$ we obtain the fourteen-order method with truncation error $t_i=-\frac{401234609}{29001931435587740}h^{18}u_i^{(18)}+O(h^{20})$.

To obtain unique solution for the system (5) we need six more equations. we define the following identities:

$$\begin{cases}\sum_{i=0}^{5}a_i'u_i+d'h^2u_0^{(2)}-h^4\sum_{i=0}^{13}b_i'u_i^{(4)}-t_1h^{18}u_0^{(18)}=0,\\ \sum_{i=0}^{6}a_i''u_i+d''h^2u_0^{(2)}-h^4\sum_{i=0}^{13}b_i''u_i^{(4)}-t_2h^{18}u_0^{(18)}=0,\\ \sum_{i=0}^{7}a_i'''u_i+d'''h^2u_0^{(2)}-h^4\sum_{i=0}^{13}b_i'''u_i^{(4)}-t_3h^{18}u_0^{(18)}=0,\\ \sum_{i=0}^{7}a_i'''u_{n-i}+d'''h^2u_n^{(2)}-h^4\sum_{i=0}^{13}b_i'''u_{n-i}^{(4)}-t_{n-3}h^{18}u_n^{(18)}=0,\\ \sum_{i=0}^{6}a_i''u_{n-i}+d''h^2u_n^{(2)}-h^4\sum_{i=0}^{13}b_i''u_{n-i}^{(4)}-t_{n-2}h^{18}u_n^{(18)}=0,\\ \sum_{i=0}^{5}a_i'u_{n-i}+d'h^2u_n^{(2)}-h^4\sum_{i=0}^{13}b_i'u_{n-i}^{(4)}-t_{n-1}h^{18}u_n^{(18)}=0,\end{cases}\qquad(8)$$

where all of the coefficients are arbitrary parameters to be determined. In order using Taylor's expansion obtain the fourteen-order method we find that:

$(a_0',a_1',a_2',a_3',a_4',a_5',d')=(-104,222,-108,-33,221,65)$, $(a_0'',a_1'',a_2'',a_3'',a_4'',a_5'',a_6'',d'')=(14,-108,189,-86,-3222,1,26)$,

$(a_0''',a_1''',a_2''',a_3''',a_4''',a_5''',a_6''',a_7''',d''')=(14,-108,189,-86,-3222,1,26)$,

$(b_0',b_1',b_2',...,b_{13}')=(\frac{21606087985477}{6276836966400},\frac{1149799956373}{19020718080},-\frac{5731547553079}{348713164800},\frac{90551633060431}{784604620800},-\frac{12254916211861}{59779399680},\frac{102139993550377}{348713164800},$

$\frac{33821362722527}{104613949440},\frac{2002843813681}{7264857600},-\frac{125667667228669}{697426329600},\frac{55669960874159}{627683696640},-\frac{33312548265013}{1046139494400},\frac{19621140967}{2490808320},\frac{7539889997027}{6276836966400})$,

$(b_0'',b_1'',b_2'',...,b_{13}'')=(\frac{21872086635283}{15692092416000},\frac{17234762713697}{523069747200},\frac{6523336140259}{174356582400},\frac{25674622691429}{392302310400},-\frac{6092225050271}{95103590400},\frac{12429339818809}{124540416000},$

$-\frac{28995834721193}{261534873600},\frac{6278835293}{66044160},-\frac{21756671474207}{348713164800},\frac{48341063514329}{1569209241600},-\frac{28998269075107}{2615348736000},\frac{119798004967}{43589145600},-\frac{188206711351}{448345497600},\frac{1417024681}{47551795200})$,

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$$(b_0''',b_1''',b_2''',\ldots,b_{13}''')=(\frac{2036981807401}{31384184832000},\frac{5153532945299}{1046139494400},\frac{1440948837983}{49816166400},\frac{40894572957431}{784604620800},\frac{4579549330421}{160944537600},\frac{918120066551}{158505984000},$$

$$-\frac{43079797441}{47551795200},\frac{2305017271}{2421619200},-\frac{452142431501}{697426329600},\frac{1627758359}{4926873600},-\frac{638156243929}{5230697472000},\frac{897454783}{29059430400},-\frac{6026165111}{1255367393280},\frac{32889931}{95103590400}),$$

and

$$(t_1=t_{n-1}=-\frac{751522003987}{9700566220800},\ t_2=t_{n-2}=-\frac{7343819831591}{266765571072000},\ t_3=t_{n-3}=\frac{202308444617}{533531142144000},)$$

3 Tension spline solution

The method (5) along with boundary condition (8) when we ignore the truncation errors in (7) give a system of linear equations.

Considering $U=[u_1,u_2,\ldots,u_n]^T$ and $C=[c_1,c_2,\ldots,c_n]^T$, This system can be written the following matrix equation:

$$(A+h^4BF)U=C$$

where

$$A=\begin{pmatrix}
a_1' & a_2' & a_3' & a_4' & a_5' & & & & & & & \\
a_1'' & a_2'' & a_3'' & a_4'' & a_5'' & a_6'' & & & & & & \\
a_1''' & a_2''' & a_3''' & a_4''' & a_5''' & a_6''' & a_7''' & & & & & \\
\beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 & & & & \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 & & & \\
 & . & & & . & & & . & & . & & \\
 & & . & & & . & & & . & & . & \\
 & & & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\
 & & & & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_4 & \beta_3 & \beta_2 \\
 & & & & & a_7''' & a_6''' & a_5''' & a_4''' & a_3''' & a_2''' & a_1''' \\
 & & & & & & a_6'' & a_5'' & a_4'' & a_3'' & a_2'' & a_1'' \\
 & & & & & & & a_5' & a_4' & a_3' & a_2' & a_1'
\end{pmatrix}$$

and

$$B=\begin{pmatrix}
b_1' & b_2' & b_3' & . & . & . & b_{13}' & & & & & \\
b_1'' & b_2'' & b_3'' & . & . & . & b_{13}'' & & & & & \\
b_1''' & b_2''' & b_3''' & . & . & . & b_{13}''' & & & & & \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & & \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & \\
 & . & & & . & & & . & & . & & \\
 & & . & & & . & & & . & & . & \\
 & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\
 & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\
 & & & & & b_{13}''' & . & . & . & b_3''' & b_2''' & b_1''' \\
 & & & & & b_3'' & . & . & . & b_3'' & b_2'' & b_1'' \\
 & & & & & b_3' & . & . & . & b_3' & b_2' & b_1'
\end{pmatrix}$$

$F(U)=diag(f(x_i,u_i))$, $i=1,2,3,\ldots,n-1.$

The vector C is defined by

$c_1=-\varepsilon a_0' u_0-\varepsilon d' h^2 u_0''+h^4 b_0' u_0^{(4)}$,

$c_2=-\varepsilon a_0'' u_0-\varepsilon d'' h^2 u_0''+h^4 b_0'' u_0^{(4)}$,

$c_3=-\varepsilon a_0''' u_0-\varepsilon d''' h^2 u_0''+h^4 b_0''' u_0^{(4)}$,

$c_4=-\varepsilon\beta_1 u_0+h^4\alpha_1 u_0^{(4)}$,

$c_5=0$,

.

.

$c_{n-5}=0$,

$c_{n-4}=-\varepsilon\beta_1 u_n+h^4\alpha_1 u_n^{(4)}$,

$c_{n-3}=-\varepsilon a_0''' u_n-\varepsilon d''' h^2 u_n''+h^4 b_0''' u_n^{(4)}$,

$c_{n-2}=-\varepsilon a_0'' u_n-\varepsilon d'' h^2 u_n''+h^4 b_0'' u_n^{(4)}$,

$c_{n-1}=-\varepsilon a_0' u_n-\varepsilon d' h^2 u_n''+h^4 b_0' u_n^{(4)}$.

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NUMERICAL RESULTS

In this section the presented method are applied to the following test problems if choosing

$$\alpha_1 = \frac{-94699819}{26826813405}, \alpha_2 = -\frac{22043001088}{26826813405}, \alpha_3 = -\frac{41159857396}{3832401915}, \alpha_4 = -\frac{171489887488}{3832401915}, \alpha_5 = -\frac{55281435446}{766480383},$$

$$\beta_1 = \frac{448760563}{85164487}, \beta_2 = \frac{1891726336}{85164487}, \beta_3 = \frac{-4591530956}{85164487}, \beta_4 = -\frac{5839582208}{85164487} \text{ and } \beta_5 = 190$$

we obtained the method of order $O(h^{14})$ respectively.

Example 1. Consider the following singularly perturbed boundary value problem

$$-\varepsilon u^{(4)}(x) + u(x) = -\varepsilon x((x-1)^4 x^4 - 24 - \varepsilon(5 - 60x + 210x^2 - 280x^3 + 126x^4)), \quad -1 \le x \le 1$$

with boundary conditions

$$u(-1) = -16\varepsilon, u(1) = 0 \; and \; u''(-1) = -688\varepsilon, u''(1) = 0$$

The exact solution for this problem is $u(x) = \varepsilon x^5 (1-x)^4$. The observed maximum absolute errors for different values of ε and n are tabulated in Tables 1-2 and compared with the methods in [15].

Table 1: Maximum absolute errors of Example 1 (Our fourteen-order method)

n	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$	$\varepsilon = \frac{1}{256}$
20	3.58×10^{-83}	1.62×10^{-83}	1.97×10^{-83}	3.72×10^{-84}	2.07×10^{-83}
40	1.78×10^{-82}	1.84×10^{-83}	5.03×10^{-83}	6.16×10^{-83}	2.54×10^{-83}
80	2.59×10^{-82}	9.95×10^{-83}	4.93×10^{-83}	4.28×10^{-83}	3.56×10^{-83}
160	3.32×10^{-82}	1.09×10^{-82}	5.49×10^{-83}	3.32×10^{-83}	3.52×10^{-83}

Example 2. Consider the following singularly perturbed boundary value problem

$$-\varepsilon u^{(4)}(x) + u(x) = (x-1)^4 x^8 \sin(\varepsilon x) - \varepsilon x^4(-16\varepsilon^3 (x-1)^3 x^3 (3x-2)\cos(\varepsilon x) + 96\varepsilon x(14 - 84x +$$
$$180x^2 - 165x^3 + 55x^4)\cos(\varepsilon x) + \varepsilon^4 (x-1)^4 x^4 \sin(\varepsilon x) - 24\varepsilon^2 (x-1)^2 x^2 (14 - 44x + 33x^2)\sin(\varepsilon x) +$$
$$24(70 - 504x + 1260x^2 - 1320x^3 + 495x^4)\sin(\varepsilon x)), \quad 0 \le x \le 1$$

with boundary conditions

$$u(0) = u(1) = 0 \; and \; u''(0) = u''(1) = 0$$

The exact solution for this problem is $u(x) = (1-x)^4 x^8 \sin(\varepsilon x)$. [15]
The observed maximum absolute errors for different values of ε and n are tabulated in Tables 3.

Table 2: Maximum absolute errors of Example 2 (Our fourteen-order method)

n	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$	$\varepsilon = \frac{1}{256}$
20	2.36×10^{-19}	2.31×10^{-21}	3.58×10^{-23}	3.11×10^{-25}	4.93×10^{-28}
40	3.68×10^{-24}	3.57×10^{-26}	5.47×10^{-28}	4.76×10^{-30}	7.94×10^{-33}
80	5.69×10^{-29}	5.50×10^{-31}	8.37×10^{-33}	7.29×10^{-35}	1.25×10^{-37}
160	8.75×10^{-34}	8.45×10^{-36}	1.28×10^{-37}	1.11×10^{-39}	1.96×10^{-42}

CONCLUSION

The new methods of orders 2, 4, 6, 8, 10,12 and 14 based on tension spline are developed for the solution of fourth order singularly perturbed boundary-value problems. Tables 1-2 shows that our method is better in the sense of accuracy and applicability.

CONFLICT OF INTEREST

Authors declare no conflict of interest.

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